Correction Model Resit Exam
Linear Algebra 2, June 25, 2021

1. a) $\langle A, B\rangle = tr(A^TB)$ defines an inner product on $\mathbb{R}^{2\times2}$. First note that if $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{pmatrix}$ then we have

$$tr(A^TB) = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$$

Next we check the three axioms of inner product:

(i) $\langle A, A\rangle \geq 0$ for all $A \in \mathbb{R}^{2\times2}$:

$$\langle A, A\rangle = a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 \geq 0$$

In addition: $\langle A, A\rangle = 0 \Leftrightarrow a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 = 0$

$\Leftrightarrow a_{11} = a_{21} = a_{12} = a_{22} = 0 \Leftrightarrow A = 0.$

(ii) $$\langle A, B\rangle = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22} =$$
$$b_{11}a_{11} + b_{12}a_{12} + b_{21}a_{21} + b_{22}a_{22}$$
$$= \langle B, A\rangle.$$

(iii) $$\langle A+B, C\rangle = (a_{11}+b_{11})c_{11} + (a_{12}+b_{12})c_{12} + (a_{21}+b_{21})c_{21} + (a_{22}+b_{22})c_{22}$$
$$= a_{11}c_{11} + a_{12}c_{12} + a_{21}c_{21} + a_{22}c_{22} + b_{11}c_{11} + b_{12}c_{12} + b_{21}c_{21} + b_{22}c_{22}$$
$$= \langle A, C\rangle + \langle B, C\rangle.$$

Of course, alternative answers are possible.. For example: for

(ii) $\langle A, B\rangle = tr(A^TB) = tr(B^TA) = \langle B, A\rangle$

because $\operatorname{tr}(A^T B) = \operatorname{tr}(A^T B)^T = \operatorname{tr} B^T A$

(iii) $\langle A+B, C\rangle = \operatorname{tr}(A+B)^T C = \operatorname{tr}(A^T C + B^T C)$

$= \operatorname{tr} A^T C + \operatorname{tr} B^T C$

$= \langle A, C\rangle + \langle B, C\rangle.$

b) $S = \operatorname{span}(M_1, M_2)$. First note that M_1, M_2 are linearly independent, so $\{M_1, M_2\}$ is a basis of S. By Gram-Schmidt we compute an orthonormal basis.:

$\|M_1\|^2 = \operatorname{tr} M_1^T M_1 = 1.1 = 1 \Rightarrow \|M_1\| = 1$, so we take $U_1 := M_1$

Next, the projection of M_2 onto $\operatorname{span}(U_1)$:

$$P_1 = \langle M_2, U_1\rangle . U_1$$

$$= 1. U_1 = U_1$$

Then $M_2 - P_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

Take $U_2 := \dfrac{M_2 - P_1}{\|M_2 - P_1\|} = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

Then $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \dfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}$ orthonormal basis of S

c) The least squares approximation of $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ within S is equal to

$$M^* = \langle M, U_1 \rangle . U_1 + \langle M, U_2 \rangle . U_2$$
$$= a . \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}} (b+d) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} a & \frac{1}{2}(b+d) \\ 0 & \frac{1}{2}(b+d) \end{pmatrix}$$

2) $x \in \mathbb{R}^n$ unequal to 0.. $M := x . x^T$

a) Claim: $\{x\}$ is a basis of $R(M)$.

Proof: $R(M)$ is equal to the span of the columns of M, so let's compute these columns. Let $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$. Then

$$x . x^T = \underbrace{x}_{n \times 1} . \underbrace{(x_1 \; x_2 \cdots x_n)}_{1 \times n}$$
$$= (x\, x_1 \quad x . x_2 \quad \cdots \quad x . x_n)$$
$$= (x_1 . x \quad x_2 . x \quad \cdots \quad x_n . x)$$

So $M = (x . x_1 \quad x_2 . x \quad \cdots \quad x_n\, x)$, in other words, the columns of M are multiples of the vector x, so clearly

$$R(M) = \text{Span}(x)$$

Since $x \neq 0$, this shows that $\{x\}$ is a basis.

b) $\mathrm{rank}(M) = \dim R(M) = 1$

c) Take the vector x. and compute:

$$M.x = x\underbrace{x^T.x}_{\text{this is the number } x_1^2+x_2^2+\dots+x_n^2} = (x_1^2+x_2^2+\dots+x_n^2).x.$$

So x is eigenvector with eigenvalue $x_1^2+x_2^2+\dots x_n^2$

d) We already saw that $\lambda_1 = x_1^2+x_2^2+\dots+x_n^2$ is a (nonzero) eigenvalue.

Claim: $\lambda_2 = 0$ is an eigenvalue with algebraic multiplicity $n-1$, so all eigenvalues are

Proof: $\lambda_2=0$ is certainly an eigenvalue because M is singular since $\mathrm{rank}(M) = 1 < n$.

The geometric multiplicity of $\lambda_2 = 0$ is equal to

$$g_2 = \dim N(M-0.I)$$

$$= \dim N(M) = n - \mathrm{rank}(M) = n-1$$

Hence also the algebraic multiplicity of $\lambda_2=0$ must be equal to $n-1$ (since algebraic multiplicity $\geq$ geometric multiplicity)

e) Again write $\lambda_1 := x_1^2+x_2^2+\dots+x_n^2$

The eigenvalues of M are λ_1 and 0

There exists an orthogonal matrix Q s.t.

$$M = Q \begin{pmatrix} \lambda_1 & 0 & \cdots & \\ & & \ddots & \\ & & & 0 \end{pmatrix} Q^T$$

Define $S := Q$. Then

$$M = S \begin{pmatrix} \lambda_1 & 0 & \cdots \\ & \ddots & \\ & & 0 \end{pmatrix} S^{-1}$$

so the Jordan Form is $J = \begin{pmatrix} \lambda_1 & 0 & \cdots & \\ & & \ddots & \\ & & & 0 \end{pmatrix}$.

3) a) Let $X > 0$ be a solution. Let λ be an eigenvalue of A with eigenvector $v \ (\neq 0)$

Then $Av = \lambda v$. Also

$$(Av)^H = (\lambda v)^H$$

so
$$v^H A^T = \bar{\lambda} v^H$$

Since $X - A^T X A > 0$ we must have

$$v^H (X - A^T X A) v > 0$$

so $v^H X v - \bar{\lambda} v^H X \lambda v > 0$

so $v^H X v - |\lambda|^2 v^H X v > 0$

This yields $(1 - |\lambda|^2) v^H X v > 0$.

Since $v^H X v > 0$ we must have $1 - |\lambda|^2 > 0$

so $|\lambda|^2 < 1$ and $|\lambda| < 1$.

b) $X := \sum_{k=0}^{\infty} (A^T)^k A^k$ converges.

To show that $X - A^T X A = I$. This is quite easy:

$$A^T X A = A^T \left(\sum_{k=0}^{\infty} (A^T)^k A^k \right) A$$

$$= A^T \left(I + A^T A + (A^T)^2 A^2 + \cdots \right) A$$

$$= A^T A + (A^T)^2 A^2 + \cdots$$

$$= \sum_{k=0}^{\infty} (A^T)^k A^k - I = X - I$$

so $A^T X A - X = -I$.

c) $X = \sum_{k=0}^{\infty} (A^T)^k A^k > 0$.

Proof: Let $v \neq 0$ be arbitrary. Then

$$v^T X v = \sum_{k=0}^{\infty} v^T (A^T)^k A^k v$$

$$= \sum_{k=0}^{\infty} (A^k v)^T A^k v$$

$$= \sum_{k=0}^{\infty} \| A^k v \|^2 = \| v \|^2 + \| A v \|^2 + \cdots$$

$$\geq \| v \|^2 > 0 \quad \text{since } v \neq 0.$$

(d) If $X - A^T X A > 0$ has a solution $X > 0$ then $|\lambda| < 1$ for all eigenvalues λ of A, this was proven in part a).

Conversely, if $|\lambda| < 1$ for all eigenvalues λ of A then $X - A^T X A = I$ has a solution $X > 0$. This was proven in parts b) and c).

Since $X - A^T X A = I$ and $I > 0$ we have $X - A^T X A > 0$.

4)

$$A = \begin{pmatrix} 0 & 1 & 0 \\ a & b & c \\ 0 & 0 & 1 \end{pmatrix} \qquad a, b, c \in \mathbb{R}$$

a) $\det(A - sI) = \det \begin{pmatrix} -s & 1 & 0 \\ a & b-s & c \\ 0 & 0 & 1-s \end{pmatrix}$

$$= -s(b-s)(1-s) - a(1-s)$$

$$= -s(b - bs - s + s^2) - a + as$$

$$= -bs + bs^2 + s^2 - s^3 - a + as$$

$$= -s^3 + (b+1)s^2 + (a-b)s - a$$

b) By Cayley-Hamilton we have

$$-A^3 + (b+1)A^2 + (a-b)A - aI = 0$$

By taking $a = 1$ and $b = 1$ we obtain

$$-A^3 + 2A^2 - I = 0 \quad \text{so}$$

$$2A^2 = A^3 + I$$

c) Take $a = b = 0$. Then

$$-A^3 + A^2 = 0 \quad \text{so} \quad A^3 = A^2$$

Then also $A^{12} = A^8$.

5. $A = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & a & 1 \end{pmatrix} \quad a, b \in \mathbb{C}$

a) One sees immediately that A has one eigenvalue $\lambda = 1$ with algebraic multiplicity equal to 3, for all $a, b \in \mathbb{C}$.

b) For A to be in Jordan Form we need to have $a = 0$ and $b = 0$

c) Assume $a \neq 0$. Denote by g the geometric multiplicity of eigenvalue $\lambda = 1$

$$g = \dim N(A - 1 \cdot I)$$
$$= \dim N \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & a & 0 \end{pmatrix}$$
$$= 3 - \operatorname{rank} \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & a & 0 \end{pmatrix}$$

Since $a \neq 0$, the rank equals 2 so $g = 1$.

d) $a=0,\ b\neq 0$

$$g = \dim \mathcal{N}(A-1I)$$

$$= 3 - \operatorname{rank}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} = 2$$

since $b\neq 0$.

e)

$a\neq 0$: in this case $g=1$, so there is one Jordan block, hence

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$a=0$: two possibities:

(i) $b=0$. In that case A was already in Jordan Form

$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(ii) $b\neq 0$. In this case two block, one of size 1, one of size 2 :

$$J = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(or: $J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$)